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The Texas Mathematics Teachers' Bulletin

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The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

Sam Houston

Cultivated mind is the guardian genius of democracy. . . . It is the only dictator that freemen acknowledge and the only security that freemen desire.

Mirabeau B. Lamar

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The Texas Mathematics Teachers' Bulletin

Volume X, No. 2

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CONTENTS

ELEMENTARY THEORY ON PENSIONS

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Certain problems in pensions lead to highly complicated formulas, notably problems connected with compensation for injured workmen. A whole family is thus sometimes placed under protection, with benefits varying with the number of children and other dependents, subject to change with remarriage of the widow, and conditioned by a fixed maximum total of benefit payable. Even a pension payable to just one individual may present some difficulties if the pension becomes payable when any one of a set of conditions is satisfied, the usual conditions referring to attained age, length of service, and disability.

In this paper, however, only the most elementary theory of pensions will be considered. A pension will be postulated payable to a single individual and starting at some specific age. The consideration of such a pension involves hardly more than a little arithmetic.

For computing the values of pensions, the mortality tables commonly used in this country are the McClintock Annuitant Tables—The Male Mortality Table and the Female Mortality Table. The closing entries of the Male Table are as follows:

Age x	Living l_x	Dying d_x
100	108	60
101	48	29
102	19	12
103	7	5
104	2	1
105	1	1
106	0	

Mortality tables are constructed to represent as nearly as possible average experience. The foregoing table indicates that of 108 men alive at age 100 about 60 on the average die during the course of a year, leaving 48 men alive at 101. The whole group becomes extinct before reaching age 106.

Of course, pensions do not usually start at such an advanced age as 100. But the theory can be more easily developed by merely considering a few ages, thus starting at a late age. The sum of the numbers in the l_x column, correspondig to ages 101 to 106 inclusive is 77. This divided by 108, the number of individuals alive at age 100, gives the quotient *.713*, called the *curtate expectation* of life at age 100. On the assumption that deaths will all occur the first day after the individuals reach the tabular age, the table indicates that the group 108 individuals live, collectively considered, 77 years of life—thus averaging .713 years apiece. But, on the assumption that deaths will occur uniformly throughout the year, the individuals of the group will on the average live a half-year longer; and thus, .713+.5 years or *1.213* years is called the *complete expectation of life* at age 100, as obtained from the McClintock Male Table.

Let us now interpret these results in terms of money. Suppose that an employer makes an agreemeent with 108 men of age 100 to pay $1,000 annually at the end of each year to each survivor. The employer assumes thereby an expected liability of $77,000, a liability of $713 a head. If the employer agrees also to *complete* the pension at the time of death by a payment of a fraction of $1,000 corresponding to the fraction of the last year lived, this requires an additional $500 on the average—assuming a uniform distribution of deaths during a year—and the company's liability becomes $1,213 a head. Likewise, if an employer pays a monthly pension of $83.33, payable on a fixed day each month, the employer should expect to pay on the average a total of $1,213 to a pensioner who has just reached his one-hundredth birthday.

The method of determining the expected liability of an employer who promises pensions to employers has just been considered. But, from an insurance company a pension, or *life annuity* as it is called, can be purchased by a cash payment. To obtain the *net single premium*—which together with some charge for expenses is the purchase price of the pension—the several expected payments to pensioners must be discounted at some assumed interest rate, and this leads to the consideration of compound interest.

To add 4 per cent is equivalent to taking 104 per cent of that number—it is equivalent to multiplying the given number by 1.04. The amount at 4 per cent compound interest for any number of years is obtained simply by using repeatedly this multiplier 1.04. Thus, the amount of $100 at compound interest for two years is

$$\$100(1.04)\,(1.04)=\$100(1.04)^2=\$108.16$$

The powers of 1.04 and of other *accumulation factors* $1+i$ are tabulated for various interest rates i. Likewise their reciprocals, the *discounting factors,* are tabulated. For example,

$$\frac{1}{(1.04)^2}=(1.04)^{-2}=0.92455621.$$

From this we learn that $10,000 due in two years is now worth $9,245.56, if 4 per cent compound interest is assumed. Check:

Principal	$9,245.56
Interest at 4 per cent	369.82
Amount after one year	9,615.38
Interest on this amount	384.62
Amount after two years	10,000.00

Computations for annuities usually assume that the yearly payment of $1 or one unit of money. Decimal places in sufficient number are retained so that the cash value of annuities commonly desired can be obtained with the desired accuracy by a single multiplication. With this assumption, the numbers in the l_x column of a mortality table represent the number of dollars which the company must plan to pay. Each such sum of money is to be discounted for the number of years which will elapse before it becomes payable The computation for the cash value to a man of age 100 of $1 payable to him at the end of each year that he survives can be arranged as follows. We first find the cash value for the whole group of 108 men, assuming some rate of interest—say 4 per cent—thus:

Year	Sum Payable	Factor for Discounting	Discounted Sum
1	$48	.9615385	$46.1538480
2	19	.9245562	17.5665678
3	7	.8889964	6,2229748
4	2	.8548042	1.7096084
5	1	.8219271	.8219271
			72.4749261

The value of the annuity to each man of the group is found by dividing the above result by 108. This gives .671064. Thus, on the basis of McClintock—4 per cent, the value to a man of age 100 of a $1,000 annual pension is $671.06. This is commonly raised to $1,171.06 if a *completing* payment is due—assuming a uniform distribution of deaths—or if monthly payments of $83.33 are provided, with first payment on the first of the month after the annuitant reaches the age of 100. Instead of assuming a uniform distribution of deaths, more delicate adjustments can be made by the use of Finite Differences or the Makeham formula for l_x,

$$l_x = k\, s^x\, g^n, \quad n = c^x,$$

where k, s, g, and c are constants. But such adjustments are beyond the scope of this paper.

Algebraically—we let a_x be the present value to a man of age x of an annuity of $1 payable at the end of each year that he survives, and thus

$$a_x = \frac{1}{l_x}\left(v l_{x+1} + v^2 l_{x+2} + v^3 l_{x+3} + \ldots \text{ to end of table}\right),$$

where v is the discounting factor $\frac{1}{1+i}$, i being the interest rate—in the illustration, $i = .04$. Multiplying numerator and denominator or the right by v^x, and setting

$$D_x = v^x l_x,\ D_{x+1} = v^{x+1} l_{x+1},\ \ldots\ N_x = D_{x+1} + D_{x+2} + \ldots,$$

we attain $a = \frac{N_x}{D_x}$.

The Teachers Insurance and Annuity Association, of New York City,—established by the Carnegie Corporation to provide insurance and pensions to college professors and other selected groups at cost—quotes the following rates for life annuities:

AMOUNT OF MONTHLY ANNUITY PURCHASED BY $1,000

Age at Purchase	If the Annuitant is a Man	If the Annuitant is a Woman
45	$5.71	$5.19
50	6.26	5.66
55	7.02	6.29
60	8.06	7.15
65	9.52	8.33
70	11.59	10.00

A little consideration will show that the establishment of a pension system is a major financial operation. Too often, pension schemes have been devised by men of little training in actuarial science, and have come to grief after creating expectations impossible of fulfillment. The average mind does not intuitionally grasp the magnitude of the load that a pension system imposes.

While some objections can be offered to pensions—*e. g.*, as representing deferred pay instead of increased pay—there has been demand for pensions, and this demand will probably continue. It is important that when a pension system is set up, it should be done intelligently, with adequate financial provision for performing its contracts.

COMPOSITE UNITS IN ELEMENTARY MATHEMATICS

A. A. BENNETT

It is possible and logical to confine mathematical theory to the notions concerned with pure number. We may start with the natural numbers, 1, 2, 3, . . . and study relations and properties of these numbers, introducing the language of negative numbers, fractions, irrational or complex numbers, but never losing sight of the original abstract basis of the entire subject.

In elementary instruction this abstract treatment is only partially followed. Every effort is expended in making the notion of number familiar by use of concrete examples. The kindergarten pupil learns to count the spots on a domino, to arrange four sticks in a square, and so on. We expect later on to hear problems start with such words as "if Jack has five marbles," or "if Mary has three apples." Even in the next higher grades when practice in addition and substraction has rendered the abstract concept of number reasonably familiar, the fact that $m\times n=n\times m$ is "demonstrated" by arranging a set of mn peas or other small objects in a rectangular array, and noting that when looked at in one way we have m rows of n elements each, while again we have n columns of m elements each.

But the concrete applications of number are not confined to pedagogical devices for impressing abstract truths. Not a small part of elementary arithmetic is devoted to a study of the notions of weights and meausures, their units and relations. Similarly in geometry we have lengths, areas, arcs, angles, with their respective units. To add six inches and two feet is not merely to add 6 and 2.

Euclid had an elaborate theory of proportion, a more or less distorted version of which still leaves its trace in elementary geometry. For Euclid there was a logical reality about a line that was more basic than the abstract notion of number. His axioms referred to geometrical concepts and after much discussion theorems about abstract numbers in the form of ratios were derived and utilized. To Euclid, a proportion of the form $a:b=c:d$ was not primarily a comparison of numbers, but always a comparison among four quantities, of which a and b were of like kinds and therefore comparable, while c and d were likewise comparable. Thus a and b might be both line segments, both areas, both rectilinear angles, both arcs of the same or equal circles, and so forth. Such a statement as "the product of the means is equal to the product of the extremes" would be indeed a rash remark from Euclid's point of view. For many things there would be no product possible according to Euclid. Two line sigments have a product, namely, the rectangle of which these are the sides. The study of this product is the subject of many theorems. In space a rectangle and a line segment have a product. But certainly no Euclidean product would exist for two areas, for a line segment and an angle, for two angles, and so forth. Thus, if $C:C'=AB:A'B'$, where C and C' are central angles, subtending respectively arcs AB and $A'B'$, on a common circle, then certainly neither the product of the means nor the product of the extremes can exist. One may say that although Euclid's treatment introduces pure numbers, irrational as well as rational, and in a logically satisfactory manner, the fundamental units are not abstract numbers, but are arcs, angles, lengths of line segments, areas of rectangles, and so forth.

Until the time of Descartes, despite the arithmetic notions of Diophantus and of the Indian school, Euclid's notion of extension was accepted as the unquestioned standard throughout Europe. Although the Cartesian system of analytical geometry has overthrown the despotic dominance of a geometrical unit as basis for mathematical science, physics has never had occasion to give over the ancient point of view. Rather one may say that the concept has acquired added significance. The modern physicist uses numerical relations at every phase of the study and systematization of natural phenomena. Relatively seldom however is mere number divorced from physical units to be encountered. In most equations, there is a physical unit contained in every term, and the method of counting the dimension of an equation is frequently applied as a mild check upon the accuracy of a formula. The relatively geometrical subject of mechanics in which considerations of temperature, and of optical, of electro-magnetic, and of other properties play no part, uses the physical units of time, length, and mass at every stage. In distinction from Euclid's geometrical treatment, modern physics does not balk at products and quotients whose direct spatial interpretation may not be easily obvious. Thus we have a unit of velocity expressible as a quotient of distance by time, such as one foot per second or 1 ft. / 1 sec. It is useless to insist that no real division is made and that only a symbolic quotient is involved. Mathematics is not concerned with length and time. No mathematical division takes place, since the units, foot and second are not mathematical concepts. However for physical purposes, no confusion arises when the notion of mathematical division is so extended that 1 foot is actually divided by 1 second. To say that one is unfamiliar with such a procedure is no criticism of the statement. Mathematically one can divide 6 by 3, but not 6 apples by 3, or 6 feet by 3, despite all that is said in elementary arithmetic, since the notion of apples is not based upon the system of rational numbers. In a true sense two half-apples are not equal to one apple. A clear recognition of the fact that two half-babies do not make one live infant is the basis of Solomon's famous decision. For physics however we can and do divide our foot by one second and obtain a unit of velocity. Less obvious is the unit of acceleration. There is no objection to a unit which is the square of a unit of mass, difficult as this may be to visualize. A unit which is the product of space and time, or one which is the square root of a unit of time is not unreasonable in some connections.

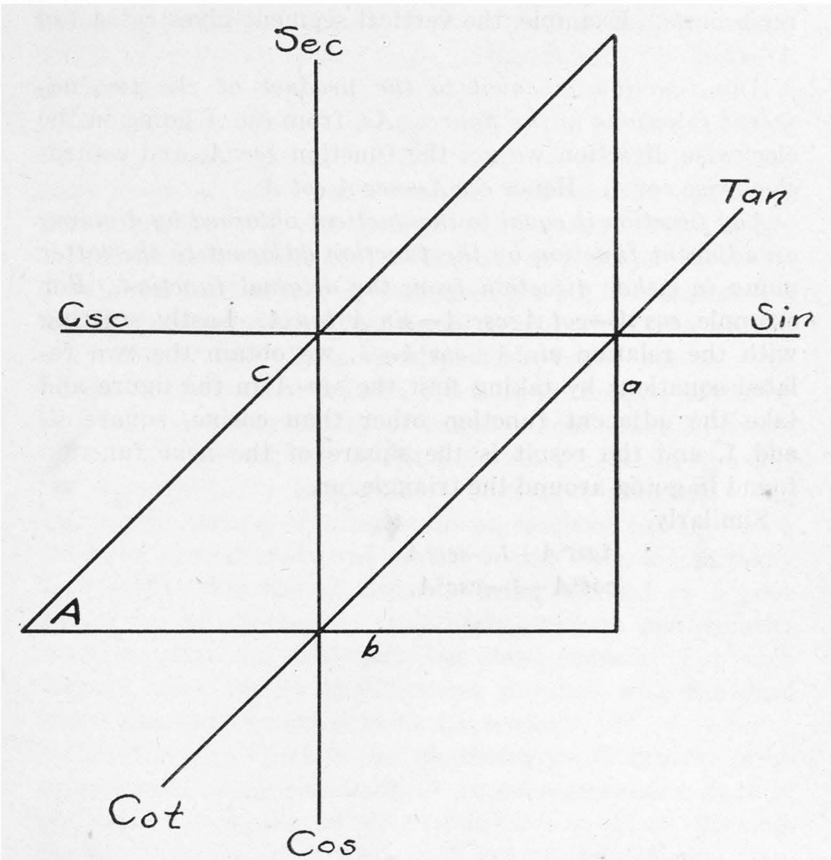
For commercial purposes we cannot specialize our results to those obtainable from mass, time, and distance. Consider the simple problem, "20 apples are to be divided evenly among 8 boys. How many apples should each boy receive?" The answer desired is in units of apples per boy, or "apples/boy," as it may be written. This answer in the form desired is obtained by dividing 20 apples by 8 boys, giving 20 apples÷8 boys=2½ apples/boy. A problem of 102 apples in three baskets gives 102 apples÷3 baskets=34 apples/basket or 34 apples per basket. From the fact that the work per day of *A*, plus the work per day of *B*, equals the combined work per day of *A* and *B*, we conclude that if *A* does a piece of work in 6 days, and *B* in 12 days, together they will do it in 4 days. Indeed (w/(6 days))+(w/(12 days))=(¼)w/days=w/(4 days). In such a problem as this "52 light bulbs were bought for $13, how much does each cost?", it is frequently baffling to the elementary pupil to know whether to multiply or to divide, and if to divide, in which order the division is to occure. Of course it *should not* be difficult, but experience shows that it often is. When the answer is emphasized as being in dollars per bulb, we see that 13 dollars÷52 bulbs, is the only way in which the right sort of answer can be obtained.

Some years ago, while I was an undergraduate, I was associated in a sophomore mathematics course with a fellow student of mature years, Mr. C., who was head of the department of mathematics at a State normal college. Mr. C. had great difficulty in keeping up with the course, and one time remarked to me upon the fact that he was right then receiving a salary considerably in excess of that which the instructor in our course was receiving, and for instruction in the same branch of learning. Upon my inquiry, Mr. C. told me that he was devoting all his energy to impressing upon the prospective teachers the distinction between division and partition, a distinction which no one at the University would have stated in the manner demanded by him at the normal school. Perhaps that fashion has passed.

What are we to say of the simple statement, 2 feet × 3 feet = 6 square feet? This is a statement to which Euclid would have subscribed, and which for 2,000 years at least, would have appeared obvious to all who were trained to think about it. This is a statement in line with all modern trends in physical science. This is the sort of statement that the younger school children are likely to make. Shall we as teachers spend our energies in emphasizing with false pedantry the lack of logic in relations of this sort whose only fault lies in the minds of those who train themselves to be blind to its appeal?

A DEVICE FOR MEMORIZING RELATIONS BETWEEN TRIGONOMETRIC FUNCTIONS

H. S. VANDIVER, UNIVERSITY OF TEXAS



It is my purpose to call attention to a scheme which is quite old, but it has not appeared in any trigonometry which I have seen during the last twenty years.

Draw a right triangle as above and consider the angle A. Any function of it is obtained by beginning at the end of *the line segment labeled with the name of the function and following the segment across the triangle.* For example, to define $sin\, A$ we consider the horizontal segment crossing the triangle beginning at the right, the first side of the triangle it crosses is a and the second c, so that $sin\, A = a/c$.

The functions situated on the two ends of a segment are reciprocals. Example, the vertical segment gives $cot\, A\; tan\, A = 1$.

Any function is equal to the product of the two adjacent functions in the figure. As, from $csc\, A$ going in the clockwise direction we get the function $sec\, A$, and contraclockwise $cot\, A$. Hence $csc\, A = sec\, A\; cot\, A$.

Any function is equal to the quotient obtained by dividing an adjacent function by the function adjacent to the latter going in either direction from the original function. For example, $cos\, A = cot\, A/csc\, A = sin\, A/tan\, A$. Lastly, starting with the relation $sin^2 A \pm cos^2 A = 1$, we obtain the two related equations by taking first the $sin\, A$ in the figure and take the adjacent function other than cosine, square it, add 1, and the result is the square of the next function found in going around the triangle, or,

Similarly,

$$tan^2 A + 1 = sec^2 A$$
$$cot^2 A + 1 = csc^2 A.$$

DO LOGARITHMS BELONG IN ALGEBRA OR IN TRIGONOMETRY?

BY ALBERT A. BENNETT, UNIVERSITY OF TEXAS

Many teachers of mathematics, being hazy as to the historical aspects of the subject they are teaching, conclude without investigation that mathematics is at least a stationary subject. It frequently happens that the teacher has no great affection for the subject. While a teacher of domestic science may be hard to find, and while such a teacher must show at least some knack in the manual details of demonstration, it is not infrequently the case that a school superintendent thinks of mathematics as a subject in which a watchful teacher merely compares the pupil's results with the answers printed in the special key to which the teacher alone has access. This view, perhaps borne out to some extent by the teacher's habit, often leads those in authority to place the responsibility of teaching mathematical topics upon whatever teacher is not adapted by special aptitude or preparation to handle less convenient subjects. The temporary decline of interest in high-school German as a result of race hatred bred by the World War, led to many a competent teacher of German being utilized as a poor teacher of mathematics. For such teachers mathematics appears not only stationary but dead indeed. For such reasons mathematics is sometimes grouped with the dead languages, and regarded as fit for burial.

Of course, the fact is that mathematics is rapidly progressing. It forms the basis of an ever-increasing part of the technical results of modern science in all its developments. It is spreading and ramifying in a bewildering manner and has long since become too comprehensive for any person to grasp in its entirety. Every half century since the year 1500 has practically doubled the body of existing mathematical knowledge. Unlike the imposing modern sciences, mathematics has preserved as still vitally significant, extensive domains of study initiated even as far back as in the height of ancient Grecian civilization. Elementary mathematics changes but little owing chiefly to the fact that it was a subject of centuries of careful reasoning even yet accepted as correct. To make a comparison, American history is surely a modern and progressive subject. American history, a hundred years from now, must of necessity treat of many topics not mentioned today. Not only will entirely new dates be considered, but recent facts will doubtless by that time be viewed from a different angle. It is to be expected that a revised judgment, in the light of subsequent developments, will affect the estimated relative importance of such presidents as McKinley, Taft, Roosevelt, Wilson, Harding, Coolidge, not to mention more important matters. Earlier topics, such as the discovery of America, the conquests of the early Spanish adventurers, the Declaration of Independence, the Louisiana Purchase, will perhaps be mentioned in approximately the same words as they are today.

Elementary arithmetic, elementary algebra, elementary geometry, are, of course, practically stationary because they are of such antiquity, and because mathematical science is so orderly in its growth. Research continues in arithmetic, in algebra, in geometry, but the newer results affect relatively slightly the usual content of a first course. And this is to be expected. The change in elementary mathematical instruction is also probably much less than it should be, on account of a conservatism due to ignorance. A teacher usually prefers to repeat what has already become familiar rather than to decide on the merits of the question among a number of new and competing plans, so that a certain part of the stationary nature of elementary mathematical instruction may be reasonably attributed not to the character of subject itself, as much as to the ignorant prejudice of those who are responsible for the choice of texts.

It has been customary to treat of logarithms in three different parts of the mathematical course of a college student. The subject is usually discussed in algebra. Certainly, since a logarithm is an exponent, the study of the laws of exponents is merely an introduction to logarithms. Practice in fractional, negative, and zero exponents prepares a natural approach to a consistent study of the entire system of a common logarithms. Sometimes the algebra text takes up the study of series, and thus introduces the exponential series in a logical manner. Again, logarithms appear in a course in trigonometry. Trigonometry very properly emphasizes two rather different disciplines. On the one hand we have the formulas and identities which follow algebraically from the original definitions. This part of trigonometry might be regarded as merely a development of certain geometrical and algebraic relations. It makes no new postulates and introduces no new methods. Whatever can be proved by means of trigonometrical identities can also be established, although usually somewhat awkwardly, without reference to trigonometric notations. No tables are used. Logically we have here geometry and algebra specialized somewhat as to topic but in nowise constituting a new branch of investigation. On the other hand, elementary trigonometry usually deals also with the numerical solution of triangles. This is connected with the handling of tables, and assumes continuity, approximate methods of solution by linear interpolation and many such notions far removed from the rational operations of algebra. Incidentally, it utilized logarithmic tables in much the same way that it uses tables of trigonometric functions.

Logarithms are introduced for a third time in connection with calculus. The theorem of the mean, and Taylor's formula, find rational illustrations in connection with exponential and trigonometric series. Many texts in calculus even discuss to a small extent the special transformed series which are found to be particularly well adapted to the computation of logarithms. They even give as exercises to student the computation of the natural logarithms of certain special numbers such as of 5 and of 7.

In discussing the question as to whether logarithms belong in algebra or in trigonometry, one might feel inclined to remark, "obviously in both." Instead of dismissing the question thus abruptly, let us inquire what one might naturally include under the term "algebra."

Our high school and college texts usually take up some forty to sixty topics and include all of them under the blanket term—algebra. The choice of topics has remained almost unaltered for well over a century, although more scholarly treatises are now showing quite a different choice of sub-topics. A number of the usual topics may be grouped under the general head "Algebraic Notation and the Simplification of Algebraic Expressions." Another general head would be "Theory of Equations" (in one unknown), another would be "Simultaneous Equations," and so forth. Under such general headings would appear numerous special topics, such as simultaneous linear equations, determinants, theory of the quadratic, graphical methods, and so on. In order to make more definite a separation between algebra and the theory of functions, it is becoming increasingly customary to emphasize "rational algebra" and to establish as a criterion the following condition: A theorem belongs to rational algebra in a domain if it employs only the rational operations of addition, subtraction, multiplication and division in this domain. Ordinarily, the domain is obtained by introducing at most a finite number of irrational roots of algebraic equation, when the equations themselves have ordinary integral coefficients. Thus any notion of an infinite process, despite its primary role in the classical theory of equations, has no place in rational algebra. While it is natural to think of rational operations among exponents as being of the same sort as rational operations among other quantities, special questions arise at once, such as the following: Is 0^0 a number, if not what is it? Is $4^{1/2}$ ambiguous, being either -2, or $+2$, or is it unique? Is $(-1)^{1/2}$ to be interpreted and studied when exponents are first introduced, or should i, the square root of minus unity, be reserved for later study? Does $8^{1/3}$ have three values or only one? If multiple valued interpretations are accepted, how may values have $1^{\sqrt{2}}$? Does a negative number have a logarithm? If not, what are we to say of the relation $(-2)^2=4$? If it does, then it must have an infinite number of distinct complex values, and a positive real quantity will then also have an infinite number of distinct logarithms. What are we to say about these? A brief examination of the question is sufficient to show that all mention of logarithms can well be omitted from rational algebra.

As already noted, the practical use of logarithmic tables must be assumed in connection with tabular work in trigonometry. What can be said as to the theory of logarithms? One may distinguish two parts in this theory, one related to the logarithms of real positive quantities and the other a study of the exponential relations among complex numbers. The notions of interpolation, of continuity, of one valued monotonic function, are also employed in the theory of the numerical approximation of roots, a branch of algebra, perhaps, but not of rational algebra. The subject is hard to divorce from the general topic of infinite series, and is strictly a part, although an elementary part, of the theory of functions of a real variable. The more general theory of exponential relations among complex quantities is bound up with De Moivre's theorem and Euler's relation between the exponential and trigonometric functions. It forms naturally a part of trigonometry.

We may conclude that the theory of real logarithms employs notions analogous to those of trigonometric tabulations and has nothing in common with rational algebra. Historically the computations of irrational roots by Newton's, Horner's, or other methods bring in analogous notions. The theory of logarithms of complex numbers is part of the general theory of trigonometric representations of complex numbers and is a branch of what one might call "higher trigonometry." In a sense, then, one would be justified in asserting that logarithms belong in Trigonometry and not in Algebra.

THE PYTHAGOREAN THEOREM

ALBERT A. BENNETT, UNIVERSITY OF TEXAS

There may be, perhaps, reasonable doubt as to who merits the honor of having first demonstrated the celebrated theorem that the sum of the squares on the two legs of a right triangle is equal to the square on the hypothenuse. Some slight claims may be made in behalf of the star gazers of Chaldea, or of the rope-stretchers of the banks of the Nile, or of the priestly temple-builders of later Egyptian dynasties. Even the Indian sages offer independent claims. But such historical conjectures we shall leave to others, and we shall be content to refer to the proposition as the Pythagorean theorem, without worrying as to the possibly mythical character of this reputed scientist or as to any other claims made by the school of the Pythagoreans.

Unlike many of the familiar propositions of geometry, this theorem does not impress most beginners as obvious. Some sort of proof is required even by the most heuristic intuitionalist. It would hardly seem incredible to the uninitiated if the theorem were found to be false, so that this shares with the other Pythagorean theorem—that of the sum of angles of a triangle—the honor of being one of the few elementary theorems of plane geometry that might properly be called "remarkable," deserving of remark. One can test with fair assurance a so-called proof. A direct tactical demonstration that the triangle with sides, 3, 4, and 5 units in length is a right triangle may be made in many ways. One can also show directly that 5, 12, 13 are possible sides for a right triangle. The inference from one or two such examples, that a triangle with sides, a, b, c, respectively, is right, whenever $a^2+b^2=c^2$ (a converse of the theorem mentioned) is not immediate. It is hard to tell whether an historical reference to a right triangle of sides, a, b, c, where $a^2+b^2=c^2$, implies anything more than the acquaintance with such isolated examples. But should there be found unquestioned documentary evidence that some long-perished writer believed and asserted the validity of the Pythagorean theorem, one might still inquire whether this was an observation based on inadequate induction, or whether, in fact, a proof was available.

Unquestionably the theorem in its generality was accepted by mathematicians preceding Euclid. Some of these, at least, had what was regarded as convincing proof. The proof usually given in school texts is copied from a demonstration admittedly due to Euclid himself. This question naturally arises: wherein was Euclid's proof an acceptable improvement upon current methods? One is inclined, perhaps, to assume that Euclid's proof must have been more simple, in the sense of being more intuitionally obvious, than its predecessors. But this assumption is not justified. To obtain the background of this and hundreds of other questions, the reader is most urgently recommended to read and study Heath's monumental *Thirteen Books of Euclid's Elements.* The few remarks here made are suggested wholly by this source.

I shall give the essential steps of two alternative proofs of the Pythagorean theorem. These have been famous for hundreds of years and have been exhibited in countless variant forms. They are thus merely typical of treatments differing essentially from each other and from Euclid's proof. Probably both will be regarded by many teachers as distinctly preferable to Euclid's own form, and yet it seems to me that this is based upon a misunderstanding of the Euclidean viewpoint.

I. PROOF BY SIMILAR FIGURE

Given: ABC a right triangle with hypothenuse AB, and with AD a perpendicular from C on straight line ADB.

To prove: $AB^2 = AC^2 + CB^2$.

1. Triangles ADB, CDB, ACB are similar, with homologous elements in the order mentioned.

2. $AD : AC = AC : AB$, by similar figures. Likewise

3. $DB:CB = CB:AB$
   Hence
4. $AD \times AB = AC^2$ Product of extremes equals product of means.
5. $DB \times AB = CB^2$
   Thus by addition
6. $(AD+DB) \times AB = AC^2 + CB^2$
   But
7. $AD + DB = AB$
   Hence
8. $AB^2 = AC^2 + CB^2$

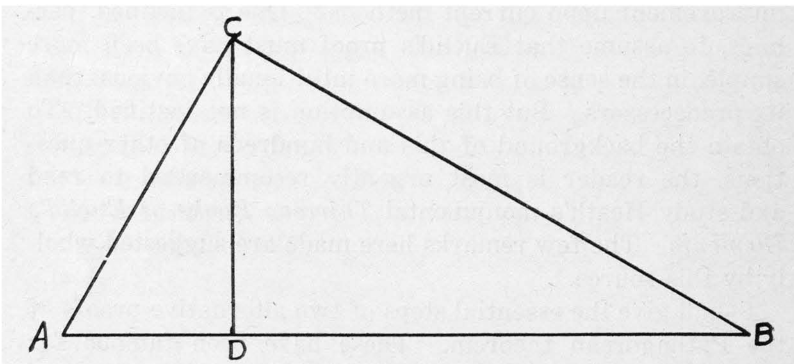


Figure 1

II. PROOF BY REPEATED FIGURES

Given: Right triangle ABC, with legs $BC=a$, $CA=b$, and hypothenuse $AB=c$.

To prove: $a^2+b^2=c^2$.

Construction: Draw. square $DFHJ$, of side $a+b$ with sides divided by points E, G, I, K, respectively, so that $DE=FG=HI=JK=a$, $EF=GH=IJ=KD=b$.

Proof:

1. $EG=GI=IK=KE=c$.
2. Area square $DFHJ=(a+b)^2=a^2+b^2+2ab$
3. Area of interior square, $EGIK=c^2$
4. Difference of areas = 4 times area of triangle $ABC= 4 \times \frac{1}{2}ab=2ab$.
5. Hence by subtraction $a^2+b^2=c^2$.

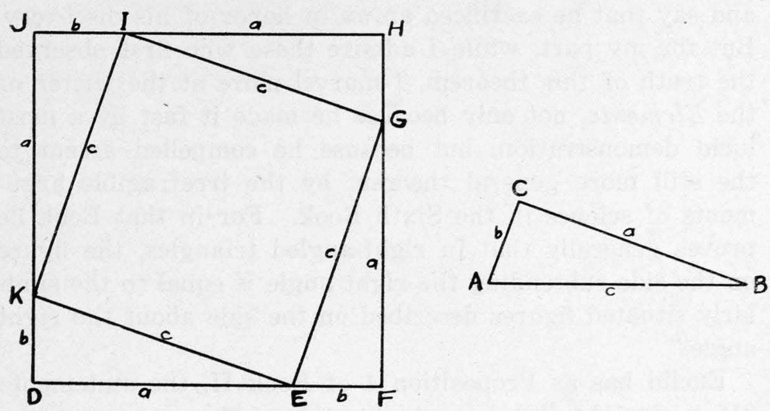


Figure 2

Now there is reason to suppose that the demonstration by similar figures was current before Euclid's time. There can be no question of its validity or of its simplicity. It furthermore embodied the very methods that Euclid himself employs. Why then did Euclid offer a more complicated discussion?

The discussion of similar figures introduces algebraic notions that Euclid regarded as sufficiently abstract to deserve being deferred to Book VI. If we could realize the state of arithmetic in the time of Euclid, and appreciate the technical difficulties in elementary computation, we, too, would marvel at Euclid's abstract accomplishment in Book VI, rather than think of much of it as cumbersome algebra. Euclid's task in writing the *Elements* was primarily one of codification. In sorting out the minimum essentials, his imagination was aroused by the character of the Pythagorean theorem. He was apparently the first to appreciate that simple as was the demonstration by similar figures, yet the theorem could be proven without any appeal to such abstract notions. Euclid's success in establishing it as Book I, Proposition 47, elicited the admiring comment of his more intelligent pupils. Proclus writes: "If we wish to listen to those who wish to recount ancient history, we may find some of them referring this theorem to Pythagoras and say that he sacrificed an ox in honor of his discovery. But for my part, while I admire those who first observed the truth of this theorem, I marvel more at the writer of the *Elements*, not only because he made it fast by a most lucid demonstration, but because he compelled assent to the still more general theorem by the irrefragible arguments of science in the Sixth Book. For in that Book he proves generally that in right-angled triangles, the figure on the side subtending the right angle is equal to the similarly situated figures described on the side about the right angle."

Euclid has as Proposition 4 of Book II, the statement: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments." This is the proposition required for the proof by "repeated figures." Use of the proof by "repeated figures" would therefore necessitate putting the Pythagorean theorem not earlier than Book II, in which geometrical algebra is developed.

It is quite reasonable to suppose that Euclid was unfamiliar with the proof by "repeated figures," but even apart from his success in placing this proposition in Book I, it is possible that he would not have valued highly this alternative method of proof. Euclid's language suggested a figure composed of a right triangle and adjacent to each side a square. Any discussion from his point of view might have presupposed this figure. The introduction of straight lines joining points of the figure is in accordance with usual methods of analysis. One might reasonable object, however, on stylistic grounds to the introduction of a wholly new figure and by implication of a third figure also, by means of which a proof is to be effected rendering apparently superfluous the original figure. The actual construction of these additional points and lines, simple as they are to describe, would render the total set of elements and relations large, if not excessive. One might say briefly that at least certain able critics would regard this suggested proof by "repeated figures" as not wholly in accordance with the trend of Greek geometrical methods.